

A Path Integral Approach to Derivative Security Pricing: I. Formalism and Analytical Results

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We use a path integral approach for solving the stochastic equations underlying the financial markets, and we show the equivalence between the path integral and the usual SDE and PDE methods. We analyze both the one-dimensional and the multi-dimensional cases, with point dependent drift and volatility, and describe a covariant formulation which allows general changes of variables. Finally we apply the method to some economic models with analytical solutions. In particular, we evaluate the expectation value of functionals which correspond to quantities of financial interest.

INTRODUCTION

The starting point of our analysis is that quantities as the stock prices, the option prices, and the interest rates satisfy differential stochastic equations (SDEs), i.e. ordinary differential equations with a superposed white noise. Such equations are called Langevin equations, and they are extensively used in the financial literature. The solutions of such equations are usually obtained by solving the associated partial differential equations (PDEs).

In this paper we want to describe an alternative approach based on the path integral formulation. The notion of path integrals, also called Wiener integrals in stochastic calculus, and Feynman integrals in quantum mechanics, is known from a long time. A proper mathematical definition of the Wiener integral can be found in the original works of Wiener [1], and Kac [2], while the quantum mechanical analogous has been introduced by Feynman [3].

The importance of this formalism lies in the possibility of employing powerful analytical and numerical techniques, developed in physics, for solving the usual problems of option pricing. Some attempts of using this approach in finance have been described in recent literature (see, for instance, Ref. [4]). Here we discuss the path integral formulation in a general manner, and, as examples, we solve some well known economical models. Furthermore, in a forthcoming paper, we will describe some numerical methods.

In section I, we show the general one-dimensional formalism. In section II, we extend the formulation to the multi-dimensional case. In section III, we discuss a covariant formulation which is necessary to perform a general transformation of variables. In section IV, we define the expectation value of a general functional. Finally, in section V, we give some analytical results.

I. TRANSITION PROBABILITY AND PATH INTEGRAL FORMALISM IN ONE DIMENSION

A. Langevin equation and discretization problem

In 1908 the french physicist Paul Langevin [5] wrote down a differential equation containing a Gaussian white noise coefficient, $f(\tau)$,

$$\dot{x}(\tau) = a(x, \tau) + \sigma(x, \tau)f(\tau). \quad (1)$$

where $x(\tau)$ is a stochastic process to be determined. Such white-noise-driven differential equations are often used in physics and chemistry and are the oldest form of SDEs. The Langevin equation written above does not define univocally a stochastic process, and it has to be supplemented with an additional interpretation rule (see, for instance, Refs. [6–8]). This is related to the ambiguity in the discretizations of this equation. We may remove this ambiguity by introducing additional structure into the Eq. (1), and discretizing the Langevin equation as

$$\Delta x = a(y + \zeta \Delta x, t) \Delta t + \sigma(y + \eta \Delta x, t) \Delta w, \quad (2)$$

where t is the initial time, Δt the time step, and $y = x(t)$. If we expand in Δx and recall that $\Delta w \sim O(\sqrt{\Delta t})$, we find, $\Delta x = \sigma(y, t) \Delta w + O(\Delta t)$, and, to the leading order in Δt ,

$$\Delta x = a(y, t) \Delta t + \eta \sigma(y, t) \frac{\partial \sigma}{\partial x}(y, t) \Delta w^2 + \sigma(y, t) \Delta w. \quad (3)$$

Now, since in the framework of stochastic calculus the following equality holds,

$$\Delta w^2 \doteq \Delta t, \quad (4)$$

(where we used the usual symbol [10], \doteq), the Eq. (3) becomes

$$\Delta x = A(y, t) \Delta t + \sigma(y, t) \Delta w, \quad (5)$$

where

$$A(x, \tau) = a(x, \tau) + \eta \sigma(x, \tau) \frac{\partial \sigma}{\partial x}(x, \tau). \quad (6)$$

The Langevin equation, written in this form, describes a well defined stochastic process, and different η 's correspond to distinct processes. For example, $\eta = 0$ and $\eta = \frac{1}{2}$ correspond to the Itô and the Stratonovich interpretations, respectively. Note that the Eq. (5) does not depend on ζ . In conclusion,

- I. *A stochastic differential equation is well defined only if both a continuous expression and a discretization rule are given.*

From now on, we will always write the underlying stochastic equation in the discretized form (5), understanding that the continuous limit must be taken. Therefore the stochastic process will be defined when the functions $A(x, \tau)$ and $\sigma(x, \tau)$ are given.

Unfortunately there is still an ambiguity. In fact, even if a stochastic process is well defined, the SDE describing such a process is not univocal. Let us consider the following family of equations, depending on the parameter, κ ,

$$\dot{x}(\tau) = a(x, \tau) - \kappa \sigma(x, \tau) \frac{\partial \sigma}{\partial x}(x, \tau) + \sigma(x, \tau) f(\tau). \quad (7)$$

By expanding in Δx , as above, we obtain

$$\Delta x = [a(y, t) + (\eta - \kappa) \sigma(y, t) \frac{\partial \sigma}{\partial x}(y, t)] \Delta t + \sigma(y, t) \Delta w. \quad (8)$$

Therefore, if we choose the appropriate discretization rule for each parameter, κ , i.e. by fixing η such that $\eta - \kappa$ is fixed, we can describe the same process by different SDEs. In other words,

- II. *Many different continuous expressions for the SDE, with the appropriate discretization rule, define the same stochastic process.*

We want to stress that all economic and financial applications of stochastic calculus have used so far the Itô prescription because it gives the coefficients of the SDE a simpler and meaningful interpretation (in particular, the drift coefficient, $a(x, \tau)$, appears directly). On the other hand, the Stratonovich prescription, for example, allows to employ the usual rules of calculus instead of the more complex Itô calculus. However we have seen above that any SDE with the Itô prescription is equivalent to some other SDE with the appropriate prescription. This more general approach is needed in order to describe the connection with the equivalent difficulties which arise in the path integral formulation, where we cannot avoid a general treatment to explain the analytical and numerical techniques of computation.

Finally, we observe that an alternative description of the stochastic process is given by the following equivalent partial differential equations, which can be derived in an unambiguous way from the Langevin equation (5) (see, for example, Ref. [9]):

- Kolmogorov's backward equation

$$\frac{\partial}{\partial t} \rho(x, T | y, t) = \left\{ A(y, t) \frac{\partial}{\partial y} + \frac{1}{2} \sigma^2(y, t) \frac{\partial^2}{\partial y^2} \right\} \rho(x, T | y, t); \quad (9)$$

- Kolmogorov's forward equation or Fokker-Planck equation

$$\frac{\partial}{\partial T} \rho(x, T | y, t) = \left\{ -\frac{\partial}{\partial x} A(x, T) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, T) \right\} \rho(x, T | y, t). \quad (10)$$

The solution of these equations is the transition probability function, $\rho(x, T | y, t)$. Note that the ambiguities of the continuous Langevin equation (1) are not present here for the following reasons: the first ambiguity is solved once the functions $A(x, \tau)$ and $\sigma(x, \tau)$ are given; the second one, which is connected to the differential operator ordering, is fixed once this ordering has been fixed.

B. Stochastic differential equation and short-time transition probability

The Brownian motion can be seen as the convolution of an infinite sequence of infinitesimal (short-time) steps. This constitutes the bridge between local equations and an integral formulation of the problem. Let us write the simple stochastic equation

$$\Delta x = \sigma \Delta w. \quad (11)$$

If w is a Wiener process this equation define a Markov process with zero average and variance equal to $\sigma^2 \Delta t$. The corresponding short-time transition probability is given by

$$\rho(x, t + \Delta t | y, t) = \sqrt{\frac{1}{2\pi \Delta t \sigma^2}} \exp \left\{ \frac{-(x - y)^2}{2 \Delta t \sigma^2} \right\}. \quad (12)$$

In general, we do not have an explicit expression for the short-time transition probability corresponding to the stochastic equation (5). However we can write the following general expression, which is correct up to $O(\Delta t)$,

$$\rho(x, t + \Delta t | y, t) \simeq \sqrt{\frac{1}{2\pi \Delta t \sigma(y, t)^2}} \exp \left\{ \frac{-(x - y - A(y, t) \Delta t)^2}{2 \Delta t \sigma(y, t)^2} \right\}, \quad (13)$$

where $A(x, \tau)$ is given in Eq. (6). A proof of this is reported for completeness in Appendix A. The equation (13) gives a prescription to write the solution of a Fokker-Planck equation in the form of a convolution product of short-time transition probability functions.

C. Finite time transition probability and path integral

The finite time transition probability can be written as the convolution of short-time transition probabilities

$$\rho(x, T | y, t) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dx_N dx_{N-1} \dots dx_1 \rho(x, T | x_N, T - \Delta t) \rho(x_N, T - \Delta t | x_{N-1}, T - 2\Delta t) \dots \rho(x_1, t + \Delta t | y, t), \quad (14)$$

with $\Delta t = \frac{T-t}{N}$. Therefore, by substituting the expression (13) in the previous equation, we get

$$\rho(x, T | y, t) \simeq \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^N dx_i \prod_{j=1}^{N+1} \frac{\sigma(x_{j-1}, t_{j-1})^{-1}}{\sqrt{2\pi \Delta t}} \exp \left\{ -S_0(x_j, t_j; x_{j-1}, t_{j-1}) \right\}, \quad (15)$$

with

$$S_0(x_j, t_j; x_{j-1}, t_{j-1}) = \frac{1}{2 \sigma(x_{j-1}, t_{j-1})^2} \left[\frac{(x_j - x_{j-1})}{\Delta t} - A(x_{j-1}, t_{j-1}) \right]^2 \Delta t \quad (16)$$

(the meaning of the subscript, 0, will be explained in the next section). We can interpret $\frac{(x_j - x_{j-1})}{\Delta t}$ as a mean velocity in the time interval Δt , and $\sigma(x_{j-1}, t_{j-1})^{-2}$ as a mass. Then a Lagrangian structure appears explicitly. Hence, the limit for $N \rightarrow \infty$ of the RHS of Eq. (15) can be formally written as

$$\rho(x, T | y, t) = \iint_{x(t)=y}^{x(T)=x} \mathcal{D}[\sigma(x, \tau)^{-1} x(\tau)] \exp \left\{ - \int_t^T L_0[x(\tau), \dot{x}(\tau); \tau] d\tau \right\}. \quad (17)$$

The functional measure, $\mathcal{D}[\sigma(x, \tau)^{-1} x(\tau)]$, means summation on all possible paths starting from $x(t) = y$ and arriving at $x(T) = x$. The integral to the exponent must be interpreted as the limit of the discrete summation for $N \rightarrow \infty$, and the function $L_0[x(\tau), \dot{x}(\tau); \tau]$ is a Lagrangian function defined by

$$L_0[x, \dot{x}; \tau] = \frac{1}{2 \sigma(x, \tau)^2} [\dot{x} - A(x, \tau)]^2. \quad (18)$$

The RHS of Eq. (17) is called “path integral”. Since in the limit $N \rightarrow \infty$ only $O(\Delta t)$ contribute to the integral, the formal expression in Eq. (17) gives the exact finite time transition probability. In conclusion, there is a complete equivalence between the differential stochastic equation, the Fokker-Planck equation and the path integral approach.

Note that in the limit $N \rightarrow \infty$ distinct expressions for the short-time transition probability, equal up to $O(\Delta t)$, give rise to the same functional expression for the path integral; therefore orders greater than $O(\Delta t)$ do not have any effects from an analytical point of view. However, if the path integral is solved numerically, a more accurate approximation for the short-time transition probability can be useful. A general procedure to obtain an approximation to any order is described in Appendix B.

D. Path integral and discretization problem

We note that the path integrals present exactly the same ambiguities of the stochastic differential equations. In particular, it is clear from its definition that the formal expression (17) is well defined only if a discretization rule, i.e. a short-time transition probability, is also given, since in general different discretization rules give rise to different results. Therefore we can say,

III. *A path integral is well defined only if both a continuous formal expression and a discretization rule are given (cf. with I).*

A further ambiguity is due to the fact that different Lagrangians, discretized by the appropriate rule, give rise to equivalent expressions for the short-time transition probability. For example let us consider the Lagrangian

$$L_{1/2}[x, \dot{x}; \tau] = L_0[x, \dot{x}; \tau] + \frac{1}{2} \partial_x A(x, \tau), \quad (19)$$

with the discretization rule given by

$$\rho(x, T | y, t) \simeq \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^N dx_i \prod_{j=1}^{N+1} \frac{\sigma(x_{j-1}, t_{j-1})^{-1}}{\sqrt{2\pi \Delta t}} \exp \{ -S_{1/2}(x_j, t_j; x_{j-1}, t_{j-1}) \}, \quad (20)$$

where

$$\begin{aligned} S_{1/2}(x_j, t_j; x_{j-1}, t_{j-1}) &= \frac{\Delta t}{2 \sigma(x_{j-1}, t_{j-1})^2} \\ &\times \left[\frac{(x_j - x_{j-1})}{\Delta t} - A\left(\frac{x_j + x_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right) \right]^2 + \frac{\Delta t}{2} \partial_x A\left(\frac{x_j + x_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right). \end{aligned} \quad (21)$$

By expanding in series of Δx and Δt up to $O(\Delta t)$, we obtain

$$\begin{aligned} S_{1/2}(x_j, t_j; x_{j-1}, t_{j-1}) &= \frac{\Delta t}{2 \sigma(x_{j-1}, t_{j-1})^2} \left[\frac{(x_j - x_{j-1})}{\Delta \tau} - A(x_{j-1}, t_{j-1}) \right]^2 \\ &+ \frac{\Delta t}{2} \partial_x A(x_{j-1}, t_{j-1}) - \frac{\Delta x^2}{2 \sigma(x_{j-1}, t_{j-1})^2} \partial_x A(x_{j-1}, t_{j-1}). \end{aligned} \quad (22)$$

Then by using the following identity, analogous to the (4),

$$\Delta x^2 \doteq \sigma(x_{j-1}, t_{j-1})^2 \Delta t, \quad (23)$$

the Eqs. (20) and (21) become equal to the Eqs. (15) and (16). The formal expression of the path integral is

$$\rho(x, T | y, t) = \iint_{x(t)=y}^{x(T)=x} \mathcal{D}[\sigma(x, \tau)^{-1} x(\tau)] \exp \left\{ - \int_t^T L_{1/2}[x(\tau), \dot{x}(\tau); \tau] d\tau \right\}. \quad (24)$$

We will call *pre-point* the path integral formulation given by the Eqs. (15), (16), (17), and (18), and *mid-point* that given by the Eqs. (19), (20), (21), and (24). In conclusion,

IV. *Many different continuous formal expressions for the path integral, with the appropriate discretization rule, define the same stochastic process* (cf. with II).

The pre-point formulation is usually simpler to perform numerical computations, while the mid-point one has some advantages to carry on analytical calculations, as we will see in the following. From now on, if not explicitly specified, we will use the mid-point formulation, but for the sake of simplicity we will omit the subscript 1/2.

E. Gauge transformation

Let us consider the case

$$L[x, \dot{x}; \tau] = \frac{1}{2\sigma^2} [\dot{x} - a(x)]^2 + \frac{1}{2} \partial_x a(x), \quad (25)$$

where we have taken $\sigma(x, \tau) = \sigma = \text{constant}$, and $A(x, t) = a(x)$. The finite time transition probability is given by

$$\begin{aligned} \rho(x, T | y, t) &= \\ &= \iint_{x(t)=y}^{x(T)=x} \mathcal{D}[\sigma^{-1} x(\tau)] \exp \left\{ - \int_t^T \left[\frac{1}{2\sigma^2} \dot{x}^2 - \frac{1}{\sigma^2} a(x) \dot{x} + \frac{1}{2\sigma^2} a(x)^2 + \frac{1}{2} \partial_x a(x) \right] d\tau \right\}. \end{aligned} \quad (26)$$

This expression contains a coupling between the derivative of the stochastic variable, \dot{x} , and the term $a(x)$. Let us perform a *gauge transformation* by introducing a function $\phi(x)$ such that

$$a(x) = \frac{d\phi}{dx}; \quad (27)$$

then we have

$$\frac{1}{\sigma^2} \int_t^T a(x) \dot{x} d\tau = \frac{1}{\sigma^2} \int_t^T \frac{d\phi}{dx} dx = \frac{1}{\sigma^2} [\phi(x) - \phi(y)], \quad (28)$$

which is the same for all paths and depends only on the end points.

It is important to note that the integrals in (28) are stochastic integrals (\dot{x} is not defined for a Brownian path), and the result (28) is true only if we use the mid-point formulation (Stratonovich prescription). In fact, if we use the pre-point formulation, the term $\frac{1}{2} \partial_x a(x)$ in Eq. (26) should be dropped, while

$$\begin{aligned} \frac{1}{\sigma^2} \int_t^T a(x) \dot{x} d\tau &= \frac{1}{\sigma^2} \int_t^T \frac{d\phi}{dx} dx \\ &= \frac{1}{\sigma^2} [\phi(x) - \phi(y)] - \frac{1}{2} \int_t^T \frac{d}{dx^2} \phi(x(\tau)) d\tau \end{aligned} \quad (29)$$

(Itô prescription). Of course the final result is the same in both cases.

The term in Eq. (28) can be put out of the path integral, and Eq. (26) becomes

$$\begin{aligned} \rho(x, T | y, t) &= e^{\Delta\phi/\sigma^2} \iint_{x(t)=y}^{x(T)=x} \mathcal{D}[\sigma^{-1} x(\tau)] \exp \left\{ - \int_t^T \left[\frac{1}{2\sigma^2} \dot{x}^2 + \frac{1}{2\sigma^2} a(x)^2 + \frac{1}{2} \partial_x a(x) \right] d\tau \right\}, \end{aligned} \quad (30)$$

where

$$\Delta\phi = \int_t^T a(x) dx. \quad (31)$$

In conclusion, the previous discussion shows that the convenience of adopting the mid-point formulation is related to the possibility of using the usual rules of integral calculus.

II. MULTI-DIMENSIONAL CASE

Until now we have discussed only the one-dimensional case. In the general case of a multi-dimensional Langevin equation in n dimensions, we have a straightforward generalization of it. Let us write the discretized Langevin equation in the following form,

$$\Delta x^\mu = A^\mu(\mathbf{x}(t), t) \Delta t + \sigma_i^\mu(\mathbf{x}(t), t) \Delta w_i \quad (32)$$

(the sum over repeated indices has been understood), where

$$A^\mu(\mathbf{x}, \tau) = a^\mu(\mathbf{x}, \tau) + \eta \sigma_i^\nu(\mathbf{x}, \tau) \frac{\partial}{\partial x^\nu} \sigma_i^\mu(\mathbf{x}, \tau). \quad (33)$$

The Eq. (32) corresponds to the following Fokker-Planck equation (in the sense that they describe the same stochastic process)

$$\frac{\partial}{\partial T} \rho(\mathbf{x}, T | \mathbf{y}, t) = \left\{ -\frac{\partial}{\partial x^\mu} A^\mu(\mathbf{x}, T) + \frac{1}{2} \frac{\partial^2}{\partial x^\mu \partial x^\nu} G^{\mu\nu}(\mathbf{x}, T) \right\} \rho(\mathbf{x}, T | \mathbf{y}, t), \quad (34)$$

where the differential operators in the right hand side act also on ρ , and

$$G^{\mu\nu}(\mathbf{x}, \tau) = \sigma_i^\mu(\mathbf{x}, \tau) \sigma_i^\nu(\mathbf{x}, \tau). \quad (35)$$

The short-time transition probability is given by

$$\rho(\mathbf{x}_j, t_j | \mathbf{x}_{j-1}, t_{j-1}) \simeq (2\pi\Delta t)^{-n/2} \sqrt{G(\mathbf{x}_{j-1}, t_{j-1})} \exp \{-S_0(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1})\} \quad (36)$$

where

$$\begin{aligned} S_0(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1}) \\ = \frac{1}{2\Delta t} G_{\mu\nu}(\mathbf{x}_{j-1}, t_{j-1}) [\Delta x^\mu - A^\mu(\mathbf{x}_{j-1}, t_{j-1})\Delta t] [\Delta x^\nu - A^\nu(\mathbf{x}_{j-1}, t_{j-1})\Delta t], \end{aligned} \quad (37)$$

and $\mathbf{x}_0 = \mathbf{y}$, $\mathbf{x}_{N+1} = \mathbf{x}$; while a path integral representation of the finite time transition probability is

$$\rho(\mathbf{x}, T | \mathbf{y}, t) = \iint_{\mathbf{x}(t)=\mathbf{y}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}[\sqrt{G} \mathbf{x}(\tau)] \exp \left\{ -\int_t^T L_0[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau] d\tau \right\}, \quad (38)$$

with

$$L_0[\mathbf{x}, \dot{\mathbf{x}}; \tau] = \frac{1}{2} G_{\mu\nu}(\mathbf{x}, \tau) [\dot{x}^\mu - A^\mu(\mathbf{x}, \tau)] [\dot{x}^\nu - A^\nu(\mathbf{x}, \tau)], \quad (39)$$

and the discretization rule given above. This expression for the transition probability corresponds to a well defined discretization procedure, i.e. the pre-point rule (cf. with Eqs. (15), (16), (17), and (18)). Another representation is given by the continuous formal expression

$$\rho(\mathbf{x}, T | \mathbf{y}, t) = \iint_{\mathbf{x}(t)=\mathbf{y}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}[\sqrt{G} \mathbf{x}(\tau)] \exp \left\{ -\int_t^T L_{1/2}[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau] d\tau \right\}, \quad (40)$$

with the Lagrangian

$$L_{1/2}[\mathbf{x}, \dot{\mathbf{x}}; \tau] = L_0[\mathbf{x}, \dot{\mathbf{x}}; \tau] + \frac{1}{2} \frac{\partial}{\partial x^\mu} A^\mu(\mathbf{x}, \tau), \quad (41)$$

and the (mid-point) discretization rule

$$\rho(\mathbf{x}_j, t_j \mid \mathbf{x}_{j-1}, t_{j-1}) \simeq (2\pi\Delta t)^{-n/2} \sqrt{G(\mathbf{x}_{j-1}, t_{j-1})} \exp \left\{ -S_{1/2}(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1}) \right\}, \quad (42)$$

where

$$\begin{aligned} S_{1/2}(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1}) &= \frac{\Delta t}{2} G_{\mu\nu}(\mathbf{x}_{j-1}, t_{j-1}) \\ &\times \left[\frac{\Delta x^\mu}{\Delta t} - A^\mu\left(\frac{\mathbf{x}_j + \mathbf{x}_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right) \right] \left[\frac{\Delta x^\nu}{\Delta t} - A^\nu\left(\frac{\mathbf{x}_j + \mathbf{x}_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right) \right] \\ &+ \frac{\Delta t}{2} \frac{\partial}{\partial x^\mu} A^\mu\left(\frac{\mathbf{x}_j + \mathbf{x}_{j-1}}{2}, \frac{t_j + t_{j-1}}{2}\right) \end{aligned} \quad (43)$$

(cf. with Eqs. (20) and (21)).

III. COVARIANT FORMULATION

The path integral formulations given in the previous sections are not covariant. As a result, a general (non linear) transformation of variables in the path integral cannot be performed according to the usual rules of calculus. In the following we describe a covariant formulation of path integrals [11,12], and we use this formulation to solve some specific problems.

The tensor, $G^{\mu\nu}$, defined in Eq. (35), transforms under a general transformation of variables as a contravariant tensor (see, for instance, Ref. [12]). If it is invertible – we will assume that this is always the case – it can be interpreted as the metric tensor of a Riemannian manifold. On the other hand, A^μ is not a contravariant vector, while the quantity

$$h^\mu(\mathbf{x}, \tau) = A^\mu(\mathbf{x}, \tau) - \frac{1}{2\sqrt{G}} \frac{\partial}{\partial x^\mu} \sqrt{G} G^{\mu\nu}(\mathbf{x}, \tau), \quad (44)$$

with

$$G \equiv G(\mathbf{x}, \tau) = \det G_{\mu\nu}(\mathbf{x}, \tau), \quad (45)$$

transforms exactly as a contravariant vector. Therefore a covariant path integral representation of the solution of Eq. (34) is given by

$$\rho(\mathbf{x}, T \mid \mathbf{y}, t) = \iint_{\mathbf{x}(t)=\mathbf{y}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}[\sqrt{G} \mathbf{x}(\tau)] \exp \left\{ - \int_t^T \mathcal{L}[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau] d\tau \right\}, \quad (46)$$

where

$$\mathcal{L}[\mathbf{x}, \dot{\mathbf{x}}; \tau] = \frac{1}{2} G_{\mu\nu} [\dot{x}^\mu - h^\mu] [\dot{x}^\nu - h^\nu] + \frac{1}{2\sqrt{G}} \frac{\partial}{\partial x^\mu} \sqrt{G} h^\mu + \frac{1}{12} R. \quad (47)$$

The scalar R is the curvature

$$R = G^{\mu\nu} R^\lambda_{\mu\lambda\nu}, \quad (48)$$

and

$$R^\lambda_{\mu\delta\nu} = \frac{\partial \Gamma^\lambda_{\mu\delta}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\delta} + \Gamma^\eta_{\mu\delta} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\delta\eta}, \quad (49)$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} G^{\delta\lambda} \left(\frac{\partial G_{\nu\delta}}{\partial x^\mu} + \frac{\partial G_{\mu\delta}}{\partial x^\nu} - \frac{\partial G_{\mu\nu}}{\partial x^\delta} \right). \quad (50)$$

As usual, the continuous formal expression (46) must be interpreted as the limit of a discretized expression

$$\rho(\mathbf{x}, T | \mathbf{y}, t) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^N d^n x_i \prod_{j=1}^{N+1} \rho(\mathbf{x}_j, t_j | \mathbf{x}_{j-1}, t_{j-1}). \quad (51)$$

In general, however, even if a discretization rule is correct in a given coordinate system, i.e. the corresponding short-time transition probability satisfies the Eq. (34) up to $O(\Delta t)$, after a transformation of variables it is not correct any more; but a covariant path integral formulation needs also a covariant discretization rule. A covariant discretization of the continuous expression (46) is given by [11,12]

$$\begin{aligned} \rho(\mathbf{x}_j, t_j | \mathbf{x}_{j-1}, t_{j-1}) &\simeq (2\pi\Delta t)^{-n/2} \sqrt{G(\mathbf{x}_j, \bar{t}_j)} \exp \left\{ -\frac{1}{2\Delta t} G_{\mu\nu}(\mathbf{x}_{j-1}, \bar{t}_j) \Delta x^\mu \Delta x^\nu \right\} \\ &\times [1 + G_{\mu\nu} h^\mu \Delta x^\nu + \Delta t (-\frac{1}{2} G_{\mu\nu} h^\mu h^\nu - \frac{1}{2\sqrt{G}} \frac{\partial}{\partial x^\mu} \sqrt{G} h^\mu - \frac{1}{12} R) \\ &+ (\frac{1}{2} (\partial_\mu G_{\lambda\nu} h^\lambda) - \frac{1}{12} R_{\mu\nu}) \Delta x^\mu \Delta x^\nu \\ &- \frac{1}{4} (\partial_\mu G_{\nu\lambda}) \Delta x^\mu \Delta x^\nu \Delta x^\lambda \\ &- (\frac{1}{12} (\partial_\mu \partial_\nu G_{\alpha\beta}) - \frac{1}{24} \Gamma_{\mu\lambda\nu} \Gamma_{\alpha}{}^\lambda{}_\beta) \Delta x^\mu \Delta x^\nu \Delta x^\alpha \Delta x^\beta / \Delta t \\ &+ \frac{1}{2} (-G_{\mu\nu} h^\mu \Delta x^\nu + \frac{1}{4} (\partial_\mu G_{\nu\lambda}) \Delta x^\mu \Delta x^\nu \Delta x^\lambda / \Delta t)^2], \end{aligned} \quad (52)$$

where $\bar{t}_j = (t_j + t_{j-1})/2$ and all functions, if it is not explicitly specified, are evaluated at \mathbf{x}_{j-1} and at any time between t_{j-1} and t_j .

With this definition of path integral, the formal measure in the expression (46) corresponds to the limit

$$\mathcal{D}[\sqrt{G} \mathbf{x}(\tau)] = \lim_{\Delta t \rightarrow 0} \sqrt{\frac{G(\mathbf{x}, \bar{t}_{N+1})}{(2\pi\Delta t)^n}} \prod_{i=1}^N \sqrt{\frac{G(\mathbf{x}_i, \bar{t}_i)}{(2\pi\Delta t)^n}} d^n x_i. \quad (53)$$

Therefore, since the quantity $\sqrt{G} d^n x$ is an invariant, after a change of variables the measure (53) transforms in the following way

$$\mathcal{D}[\sqrt{G'} \mathbf{x}'(\tau)] = \left| \frac{\partial x'_\mu}{\partial x_\nu} \right|_{\mathbf{x}, T}^{-1} \mathcal{D}[\sqrt{G} \mathbf{x}(\tau)], \quad (54)$$

where $\left| \frac{\partial x'_\mu}{\partial x_\nu} \right|_{\mathbf{x}, T}$ is the Jacobian of the transformation calculated in the final point (note that the symbol, ', does not mean derivation).

A. The case of $G^{\mu\nu} = \text{constant}$

If $G^{\mu\nu}$ is constant, the affine connections, $\Gamma_{\mu\nu}^\lambda$, and the curvature, R , are equal to zero. Then the Lagrangian (47) becomes

$$\mathcal{L}[\mathbf{x}, \dot{\mathbf{x}}; \tau] = \frac{1}{2} G_{\mu\nu} [\dot{x}^\mu - A^\mu(\mathbf{x}, \tau)] [\dot{x}^\nu - A^\nu(\mathbf{x}, \tau)] + \frac{1}{2} \frac{\partial}{\partial x^\mu} A^\mu(\mathbf{x}, \tau) \quad (55)$$

(cf. with Eq. (41)), and the discretized expression (52),

$$\begin{aligned} \rho(\mathbf{x}_j, t_j | \mathbf{x}_{j-1}, t_{j-1}) &\simeq (2\pi\Delta t)^{-n/2} \sqrt{G} \exp \left\{ -\frac{1}{2\Delta t} G_{\mu\nu} \Delta x^\mu \Delta x^\nu \right\} \\ &\times [1 + G_{\mu\nu} A^\mu \Delta x^\nu + \Delta t (-\frac{1}{2} G_{\mu\nu} A^\mu A^\nu - \frac{1}{2} \frac{\partial}{\partial x^\mu} A^\mu) \\ &+ \frac{1}{2} G_{\lambda\nu} \Delta x^\mu \Delta x^\nu \frac{\partial}{\partial x^\mu} A^\lambda + \frac{1}{2} (-G_{\mu\nu} A^\mu \Delta x^\nu)^2]. \end{aligned} \quad (56)$$

Note that the equation (56) is simply the expansion up to $O(\Delta t)$ of the expression in (42).

In conclusion, in the case of $G^{\mu\nu} = \text{constant}$, the covariant formulation becomes the mid-point one. Therefore, if we can make a change of variables which gives a constant metric (flat space), the covariant path integral can be handled by the usual methods. In section V we will show that, if the resulting Lagrangian is a quadratic form, the path integral can be solved by using well known analytical methods.

B. One-dimensional case

In the one-dimensional case the covariant formulation simplifies significantly. The transformation rule for the metric tensor in one dimension is simply

$$G'_{11} = \left(\frac{dx}{dx'} \right)^2 G_{11}, \quad (57)$$

then G'_{11} can be made equal to 1 everywhere by choosing

$$x' = \int dx \sqrt{G_{11}}. \quad (58)$$

In this coordinate system the curvature R vanishes, as follows by Eqs. (49) and (50); as a result, since R is invariant, it vanishes in every coordinate system. Then, by using the notation $G^{11} = \sigma(x, \tau)^2$, and $G_{11} = G = \frac{1}{\sigma(x, \tau)^2}$, the Lagrangian (47) becomes

$$\mathcal{L}[x, \dot{x}; \tau] = \frac{1}{2 \sigma(x, \tau)^2} [\dot{x} - h(x, \tau)]^2 + \frac{\sigma(x, \tau)}{2} \frac{\partial}{\partial x} \frac{h(x, \tau)}{\sigma(x, \tau)}, \quad (59)$$

where

$$h(x, \tau) = A(x, \tau) - \frac{1}{2} \sigma(x, \tau) \partial_x \sigma(x, \tau). \quad (60)$$

Finally, the transformation rule of the measure is

$$\mathcal{D}[\sigma'(x', \tau)^{-1} x'(\tau)] = \left| \frac{\partial x'}{\partial x} \right|_{x, T}^{-1} \mathcal{D}[\sigma(x, \tau)^{-1} x(\tau)]. \quad (61)$$

C. Itô lemma

Let us consider, for the sake of simplicity, the one-dimensional Langevin equation (5). The Itô lemma states that a function, $z(x)$, of the stochastic variable follows the process

$$\Delta z = \left(\frac{\partial z}{\partial x} A + \frac{1}{2} \frac{\partial^2 z}{\partial x^2} \sigma^2 \right) \Delta \tau + \frac{\partial z}{\partial x} \sigma \Delta w. \quad (62)$$

We can now obtain the Itô lemma from the path integral formalism.

Let us interpret the function, $z(x)$, as a change of variables from the variable, x , to the variable, z . The transformation rules for the coefficients of the Langevin equation can be obtained by

$$\sigma' = \frac{\partial z}{\partial x} \sigma, \quad (63)$$

$$h' = \frac{\partial z}{\partial x} h, \quad (64)$$

where

$$h(x, \tau) = A(x, \tau) - \frac{1}{2} \sigma(x, \tau) \partial_x \sigma(x, \tau), \quad (65)$$

$$h'(z, \tau) = A'(z, \tau) - \frac{1}{2} \sigma'(z, \tau) \partial_z \sigma'(z, \tau). \quad (66)$$

The Eq. (63) gives directly the standard deviation for the new variable, z . Moreover, by using also the relation

$$\partial_z \sigma' = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \sigma \right) = \frac{\partial x}{\partial z} \left(\frac{\partial^2 z}{\partial x^2} \sigma + \frac{\partial z}{\partial x} \partial_x \sigma \right), \quad (67)$$

we obtain

$$h' = A' - \frac{1}{2} \sigma \left(\frac{\partial^2 z}{\partial x^2} \sigma + \frac{\partial z}{\partial x} \partial_x \sigma \right). \quad (68)$$

Finally, if we compare this expression with

$$h' = \frac{\partial z}{\partial x} h = \frac{\partial z}{\partial x} \left(A - \frac{1}{2} \sigma \partial_x \sigma \right), \quad (69)$$

we obtain the drift for the variable, z ,

$$A' = \frac{\partial z}{\partial x} A + \frac{1}{2} \frac{\partial^2 z}{\partial x^2} \sigma^2. \quad (70)$$

The expressions (63) and (70) coincide with the results of the Itô lemma.

IV. EXPECTATION VALUES

The expectation value of a functional $g[\mathbf{x}(\tau); \tau]$ on the stochastic process defined by the Langevin equation (32), with fixed initial and final conditions, $\mathbf{x}(t) = \mathbf{y}$, and $\mathbf{x}(T) = \mathbf{x}$, is formally given by

$$\begin{aligned} < \mathbf{x}, T \mid g[\mathbf{x}(\tau); \tau] \mid \mathbf{y}, t >_L = \\ & \int \int_{\mathbf{x}(t)=\mathbf{y}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}[\sqrt{G} \mathbf{x}(\tau)] g[\mathbf{x}(\tau); \tau] \exp \left\{ - \int_t^T L[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau] d\tau \right\}, \end{aligned} \quad (71)$$

The discretized expression corresponding to (71) is

$$\begin{aligned} < \mathbf{x}, T \mid g[\mathbf{x}(\tau); \tau] \mid \mathbf{y}, t >_L \simeq \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^N dx_i^n \prod_{j=1}^{N+1} (2\pi \Delta t)^{-n/2} \sqrt{G(\mathbf{x}_{j-1}, t_{j-1})} \\ & \times g(\mathbf{x}_t, \mathbf{x}_1, \dots, \mathbf{x}_T; t, t_1, \dots, T) \exp \{ -S(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1}) \}, \end{aligned} \quad (72)$$

where $S(\mathbf{x}_j, t_j; \mathbf{x}_{j-1}, t_{j-1})$ represents an appropriate discretization rule, while the function $g(\mathbf{x}_t, \mathbf{x}_1, \dots, \mathbf{x}_T; t, t_1, \dots, T)$ is a discretization of the functional $g[\mathbf{x}(\tau); \tau]$. In general, different discretizations could give different results.

Obviously, the expectation value (71) can be also written in covariant form by substituting the Lagrangian, $L[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau]$, with the invariant one, $\mathcal{L}[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau); \tau]$, by using the covariant discretization rule (52), and provided that also the functional, $g[\mathbf{x}(\tau); \tau]$, is discretized in a covariant form.

A case of particular importance is when the functional can be written as

$$g[\mathbf{x}(\tau); \tau] = \exp \left\{ \int_t^T V[\mathbf{x}(\tau); \tau] d\tau \right\}. \quad (73)$$

Since the term, $V[\mathbf{x}(\tau); \tau]$, is invariant for a general transformation of variables, it can be simply included into the Lagrangian both in the non-covariant and the covariant formulations. Moreover, since such a term does not depend on the derivative of the stochastic variable, the differences among different discretizations of the functional (73) are of order larger than $O(\Delta t)$, and, therefore, it can be discretized with any rule.

A. Mean value

There are several examples of functionals. The simplest one is the expectation value of the stochastic variable x at a given time t_1 . The functional can be formally written as

$$g[x(\tau); \tau] = \int_{-\infty}^{+\infty} \delta(t_1 - \tau) x(\tau) d\tau \quad t \leq t_1 \leq T, \quad (74)$$

and the expectation value is given by

$$\langle x, T \mid x(t_1) \mid y, t \rangle_L = \int_{-\infty}^{+\infty} dz \rho(x, T \mid z, t_1) z \rho(z, t_1 \mid y, t). \quad (75)$$

For instance, in the simple case given by Eq. (11) the expectation value is easy to evaluate, and we obtain

$$\begin{aligned} \langle x, T \mid x(t_1) \mid y, t \rangle_L &= \frac{1}{\sqrt{2\pi\sigma^2(T-t_1)}} \frac{1}{\sqrt{2\pi\sigma^2(t_1-t)}} \\ &\times \int_{-\infty}^{+\infty} dz \exp\left[-\frac{(x-z)^2}{2\sigma^2(T-t_1)}\right] z \exp\left[-\frac{(y-z)^2}{2\sigma^2(t_1-t)}\right]. \end{aligned} \quad (76)$$

In absence of the integrand factor, z , we get the transition amplitude from t to T ,

$$\rho(x, T \mid y, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[-\frac{(x-y)^2}{2\sigma^2(T-t)}\right]. \quad (77)$$

If we perform the integral with the integrand factor, z , we obtain the transition probability (77) multiplied by the mean value of the functional,

$$\langle x, T \mid x(t_1) \mid y, t \rangle_L = \frac{x(t_1 - t) + y(T - t_1)}{(T - t)} \rho(x, T \mid y, t). \quad (78)$$

B. Functionals for financial quantities

Many financial quantities are defined as expectation value of functionals on a stochastic process, *with fixed initial condition*, $\mathbf{x}(t) = \mathbf{y}$. We will denote this kind of expectation value by the symbol, E_L , which is defined by

$$E_L \left[g[\mathbf{x}(\tau); \tau] \mid \mathbf{x}(t) = \mathbf{y} \right] = \int_{-\infty}^{+\infty} d^m x \langle \mathbf{x}, T \mid g[\mathbf{x}(\tau); \tau] \mid \mathbf{y}, t \rangle_L. \quad (79)$$

Note that this quantity could be formally written as a functional integral with only one extreme fixed, $\mathbf{x}(t) = \mathbf{y}$. In this case the functional measure would be completely invariant, and the whole functional integral could be formally written in full invariant form. However, from a computational point of view, this does not essentially simplify the procedure of calculation.

1. Zero-coupon bond

A first example is the evaluation of the quantity

$$P(r_t, t, T) = E_L \left[e^{-\int_t^T r(\tau) d\tau} \mid r(t) = r_t \right], \quad (80)$$

which is the price at time t of a zero-coupon discount bond with principal 1, and maturing at time T . The variable, $r(\tau)$, is the interest rate, and satisfies a SDE corresponding to the Lagrangian, $L[r(\tau), \dot{r}(\tau); \tau]$. Let us define

$$\begin{aligned}
G(r_T, T \mid r_t, t) &= \langle r_T, T \mid e^{-\int_t^T r(\tau) d\tau} \mid r_t, t \rangle_L \\
&= \iint_{r(t)=r_t}^{r(T)=r_T} \mathcal{D}[\sigma(r, \tau)^{-1} r(\tau)] \exp \left\{ - \int_t^T \tilde{L}[r(\tau), \dot{r}(\tau); \tau] d\tau \right\},
\end{aligned} \tag{81}$$

where

$$\tilde{L}[r(\tau), \dot{r}(\tau); \tau] = L[r(\tau), \dot{r}(\tau); \tau] + r(\tau). \tag{82}$$

The price is then given by

$$P(r_t, t, T) = \int_{-\infty}^{+\infty} dr_T G(r_T, T \mid r_t, t). \tag{83}$$

2. Caplet

The price of a caplet is defined by the following expectation value:

$$C(r_t, t, T, s) = E_L \left[e^{-\int_t^T r(\tau) d\tau} (\chi - P(r_T, T, s)) \theta(\chi - P(r_T, T, s)) \mid r(t) = r_t \right], \tag{84}$$

which can be seen as the price at time t of a European put option maturing at time T on a zero-coupon discount bond with principal 1, and maturing at time s , where χ is the strike price. This expression can be put in the form

$$\begin{aligned}
C(r_t, t, T, s) &= \int_{-\infty}^{+\infty} dr_T \iint_{r(t)=r_t}^{r(T)=r_T} \mathcal{D}[\sigma(r, \tau)^{-1} r(\tau)] \\
&\quad \times (\chi - P(r_T, T, s)) \theta(\chi - P(r_T, T, s)) \exp \left\{ - \int_t^T \tilde{L}[r(\tau), \dot{r}(\tau); \tau] d\tau \right\} \\
&= \int_{-\infty}^{+\infty} dr_T (\chi - P(r_T, T, s)) \theta(\chi - P(r_T, T, s)) G(r_T, T \mid r_t, t),
\end{aligned} \tag{85}$$

where $G(r_T, T \mid r_t, t)$ and $\tilde{L}[r(\tau), \dot{r}(\tau); \tau]$ have been defined in Eqs. (81) and (82), respectively. This result has been reported, and explicitly calculated for the Vasicek model, by Jamshidian (cf. Eq. (11) in Ref. [13]).

V. ANALYTICAL RESULTS

In general, the explicit analytical calculation of a path integral is a formidable task, and it is possible in a very few cases only. A class of systems which allows exact path integration is characterized by quadratic Lagrangians [14–16]. In the next sections, we will consider some cases which belong or can be reduced to this class.

A. Elementary cases

1. Harmonic Lagrangian

The harmonic Lagrangian is

$$L[x, \dot{x}; \tau] = \frac{1}{2\sigma^2} \dot{x}^2 + \frac{\omega^2}{2\sigma^2} x^2. \tag{86}$$

Note that this Lagrangian does not contain any term with a coupling between x and \dot{x} , and cannot correspond to any stochastic process, but it is the starting point for the next calculations. The solution of the path integral for this process is a well known result, and it can be cast in the form

$$I_{\text{harmonic}}(x, T | y, t) = e^{\omega(x^2 - y^2)/2\sigma^2} e^{-\omega(T-t)/2} \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \exp \left\{ -\frac{(y e^{-\omega(T-t)} - x)^2}{2\bar{\sigma}^2} \right\}, \quad (87)$$

where

$$\bar{\sigma} = \sigma \sqrt{\frac{(1 - e^{-2\omega(T-t)})}{2\omega}}. \quad (88)$$

2. Harmonic Langevin equation

Let us consider the stochastic process defined by the following SDE,

$$\Delta x = -\omega x \Delta t + \sigma \Delta w. \quad (89)$$

The corresponding Lagrangian is

$$L[x, \dot{x}; \tau] = \frac{1}{2\sigma^2} [\dot{x} + \omega x]^2 - \frac{\omega}{2}. \quad (90)$$

This Lagrangian is equal to the previous one plus a coupling between the stochastic variable, x , and its derivative, \dot{x} , and plus a constant factor. By using the prescription given in section IE, we get the following finite time transition probability,

$$\rho(x, T | y, t) = e^{-\omega(x^2 - y^2)/2\sigma^2} e^{\omega(T-t)/2} \iint_{x(t)=y}^{x(T)=x} \mathcal{D}[\sigma^{-1}x(\tau)] \exp \left\{ -\int_t^T \left[\frac{\dot{x}^2}{2\sigma^2} + \frac{\omega^2 x^2}{2\sigma^2} \right] d\tau \right\}. \quad (91)$$

The path integral in Eq. (91) is just $I_{\text{harmonic}}(x, T | y, t)$. Then, we obtain

$$\rho(x, T | y, t) = \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \exp \left\{ -\frac{(y e^{-\omega(T-t)} - x)^2}{2\bar{\sigma}^2} \right\}, \quad (92)$$

where $\bar{\sigma}$ is given by Eq. (88).

B. The Vasicek model (Ornstein-Uhlenbeck)

This model is very popular in the financial literature and it is defined by the stochastic equation [17]

$$\Delta r = a(b - r)\Delta t + \sigma \Delta w, \quad (93)$$

where the variable, r , is the short term interest rate. The finite time transition probability, which, following the notation of Ref. [13], will be denoted here by $K(r_T, T | r_t, t)$, is given by

$$K(r_T, T | r_t, t) = \iint_{r(t)=r_t}^{r(T)=r_T} \mathcal{D}[\sigma^{-1}r(\tau)] \exp \left\{ -\int_t^T L^V[r(\tau), \dot{r}(\tau); \tau] d\tau \right\}, \quad (94)$$

where

$$L^V[r, \dot{r}; \tau] = \frac{1}{2\sigma^2} [\dot{r} - a(b - r)]^2 - \frac{a}{2}. \quad (95)$$

This case is similar to that of section V A 2, with $\omega = a$, and $x \rightarrow r - b$. Then the path integral can be simply worked out by changing the variables (here we do not need the covariant formulation since this is a linear transformation of variables).

1. Expectation value

Actually, we are not really interested in the Green function for Eq. (93), but in the expectation value (81), i.e.

$$G(r_T, T | r_t, t) = \iint_{r(t)=r_t}^{r(T)=r_T} \mathcal{D}[\sigma^{-1}r(\tau)] \exp \left\{ - \int_t^T \tilde{L}^V[r(\tau), \dot{r}(\tau); \tau] d\tau \right\}, \quad (96)$$

where $\tilde{L}^V[r, \dot{r}; \tau] = L^V[r, \dot{r}; \tau] + r$. We can make the following linear change of variable,

$$z = r - b + \frac{\sigma^2}{a^2}. \quad (97)$$

The Jacobian of the transformation is equal to 1, and the Lagrangian becomes

$$\tilde{L}^V[z, \dot{z}; \tau] = \frac{1}{2\sigma^2} [\dot{z} + az]^2 - \frac{a}{2} + b - \frac{\sigma^2}{2a^2} - \frac{\dot{z}}{a}. \quad (98)$$

The last three terms can be integrated, and they give rise to a phase factor (we recall that we are using the mid-point prescription), then the integral (96) can be written as

$$G(r_T, T | r_t, t) = e^{-\Delta\theta} \iint_{z(t)=r_t-b+\sigma^2/a^2}^{z(T)=r_T-b+\sigma^2/a^2} \mathcal{D}[\sigma^{-1}z(\tau)] \exp \left\{ - \int_t^T \left[\frac{1}{2\sigma^2} (\dot{z} + az)^2 - \frac{a}{2} \right] d\tau \right\}, \quad (99)$$

where the phase factor, $\Delta\theta$, is given by

$$\Delta\theta = \int_t^T \left(b - \frac{\sigma^2}{2a^2} - \frac{\dot{z}}{a} \right) d\tau = \frac{r_T - r_t}{a} + (T - t) \left[b - \frac{\sigma^2}{2a^2} \right]. \quad (100)$$

Finally, since the remaining path integral is equal to that corresponding to the harmonic Langevin equation, by using the result (92) we obtain

$$G(r_T, T | r_t, t) = \frac{e^{(r_T - r_t)/a + (T - t)[b - \sigma^2/2a^2]}}{\sqrt{2\pi\bar{\sigma}^2}} \times \exp \left\{ - \frac{[(r_t - b + \sigma^2/a^2) e^{-a(T-t)} - (r_T - b + \sigma^2/a^2)]^2}{2\bar{\sigma}^2} \right\}, \quad (101)$$

with

$$\bar{\sigma} = \sigma \sqrt{\frac{(1 - e^{-2a(T-t)})}{2a}}. \quad (102)$$

This expression is the same that we find in the literature (cf. Eq. (13) in Ref. [13]).

2. Zero-coupon bond and caplet price in the Vasicek model

The quantity $G(r_T, T | r_t, t)$ allows to evaluate the price, $P(r_t, t, T)$, at time t of a zero-coupon discount bond with principal 1, and maturing at time T , by employing the formula (83). Since r_T appears in a quadratic form to the exponent, the integral can be performed by completing the square and using the rules for Gaussian integration. The final result is given by

$$P(r, t, T) = \exp \left[-B(T-t)r + [B(T-t) - (T-t)](b - \frac{\sigma^2}{2a^2}) - B(T-t)^2 \frac{\sigma^2}{4a} \right], \quad (103)$$

where

$$B(\tau) = \frac{1 - e^{-a\tau}}{a}. \quad (104)$$

A second task is to evaluate the caplet price given by Eq. (85). Such an expression has the structure of a truncated lognormal integration, and can be done analytically by usual methods. Here we do not discuss the details of the calculation which are given in the literature.

C. The Black-Scholes model

Let us consider the Black-Scholes model

$$\Delta S = a S \Delta \tau + \sigma S \Delta w. \quad (105)$$

where S is the stock price. Here the metric depends on the stochastic variable, S , and we need a non-linear change of variables to perform the calculation. Therefore we must use the covariant formulation.

The invariant Lagrangian corresponding to Eq. (105) can be obtained by Eq. (59), with $\sigma(S) = \sigma S$, and $h(S) = a S - \frac{\sigma^2}{2} S$. Since this is a one-dimensional problem, we can use the change of variables of Eq. (58) which gives rise to a constant metric. Then the relation between the two coordinate systems is

$$S' = \int \frac{dS}{S} = \log S. \quad (106)$$

Since the Lagrangian is invariant, we only need to exchange the quantities in the old coordinate system with the new ones, given by

$$\sigma'(S') = \sigma, \quad (107)$$

$$h'(S') = \frac{dS'}{dS} h(S) = (a - \frac{\sigma^2}{2}). \quad (108)$$

The invariant Lagrangian becomes

$$\mathcal{L}[S', \dot{S}'; \tau] = \frac{1}{2\sigma^2} [\dot{S}' - (a - \frac{\sigma^2}{2})]^2. \quad (109)$$

Furthermore, the new measure is

$$\mathcal{D}[\sigma^{-1} S'(\tau)] = S_T \mathcal{D}[\sigma(S)^{-1} S(\tau)]. \quad (110)$$

Since in the new coordinate system the invariant Lagrangian (109) coincides with the mid-point one (see, section III A), we can proceed in the usual way, and we get for the finite time transition probability,

$$\begin{aligned} \rho(S_T, T \mid S_t, t) = & e^{\Delta\phi/\sigma^2} \frac{1}{S_T} \iint_{S'(t)=\log(S_t)}^{S'(T)=\log(S_T)} \mathcal{D}[\sigma^{-1} S'(\tau)] \exp \left\{ - \int_t^T \left[\frac{1}{2\sigma^2} \dot{S}'^2 + \frac{1}{2\sigma^2} \left(a - \frac{\sigma^2}{2} \right)^2 \right] d\tau \right\}, \end{aligned} \quad (111)$$

where

$$\Delta\phi = \int_{\log(S_t)}^{\log(S_T)} (a - \frac{\sigma^2}{2}) dS' = (a - \frac{\sigma^2}{2}) [\log(S_t) - \log(S_T)]. \quad (112)$$

Finally, by using the formulae obtained previously, and collecting these expressions, we get the result of Black and Scholes.

D. The Cox Ingersoll Ross model

Another case where we need the covariant formulation of path integrals is the Cox Ingersoll Ross (CIR) model. This model has been developed to overcome a limit of the Vasicek model, i.e. the fact that the short term interest rate can take negative values. Obviously, this fact has not any real meaning. To avoid this problem Cox, Ingersoll, and Ross [18] have proposed the following process,

$$\Delta r = a(b - r) \Delta \tau + \sigma \sqrt{r} \Delta w \quad (113)$$

(for a discussion of this equation, see, for example, Ref. [19]). Here the variance is proportional to r , and goes to zero with the short term interest rate. The choice of this particular dependence was related to the solubility of the model. As we will see, in this case the path integral (81) can be expressed in a closed form through an analytical function.

Let us start from the path integral written in covariant form. The invariant Lagrangian is given by the expression (59), with $\sigma(r) = \sigma \sqrt{r}$, and $h(r) = a(b - r) - \frac{\sigma^2}{4}$. If we introduce the variable $z = 2\sqrt{r}$, we obtain

$$\sigma'(z) = \sigma, \quad (114)$$

$$h'(z) = \frac{dz}{dr} h(r) = 2 \left(\frac{a(b - z^2/4)}{z} - \frac{\sigma^2}{4z} \right). \quad (115)$$

Therefore, in the new variable, the invariant Lagrangian becomes

$$\mathcal{L}[z, \dot{z}; \tau] = \frac{1}{2\sigma^2} \left[\dot{z} - 2 \left(\frac{a(b - z^2/4)}{z} - \frac{\sigma^2}{4z} \right) \right]^2 + \frac{1}{2} \frac{\partial}{\partial z} h'(z). \quad (116)$$

In order to evaluate the expectation value (81), we must add to the Lagrangian above the term, $r = z^2/4$. Furthermore, the measure in the new variable is given by

$$\mathcal{D}[\sigma^{-1}z(\tau)] = \sqrt{r_T} \mathcal{D}[\sigma(r)^{-1}r(\tau)]. \quad (117)$$

If we now observe that the term proportional to \dot{z} gives rise to the following phase factor (see, section IE),

$$\Delta\phi = \int_{z_t}^{z_T} h'(z) dz = -\frac{a}{4}(z_T^2 - z_t^2) + (2ab - \frac{\sigma^2}{2}) \log \frac{z_T}{z_t}, \quad (118)$$

with $z_T = 2\sqrt{r_T}$, and $z_t = 2\sqrt{r_t}$, then the expectation value (81) can be written as

$$G(z_T, T | z_t, t) = \frac{2}{z_T} \exp \left\{ \frac{\Delta\phi}{\sigma^2} \right\} \iint_{z(t)=z_t}^{z(T)=z_T} \mathcal{D}[\sigma^{-1}z(\tau)] \exp \left\{ - \int_t^T L_{\text{eff}}[z, \dot{z}; \tau] d\tau \right\}, \quad (119)$$

where

$$L_{\text{eff}}[z, \dot{z}; \tau] = \frac{1}{2\sigma^2} \dot{z}^2 + c_1 + c_2 z^2 + \frac{c_3}{z^2}, \quad (120)$$

and the parameters, c_1 , c_2 , and c_3 , are given by

$$c_1 = -\frac{a^2 b}{\sigma^2}, \quad (121)$$

$$c_2 = \frac{a^2}{8\sigma^2} + \frac{1}{4}, \quad (122)$$

$$c_3 = \frac{(4ab - 3\sigma^2)(4ab - \sigma^2)}{8\sigma^2}. \quad (123)$$

The effective Lagrangian, L_{eff} , is not quadratic, but the term $\sim \frac{1}{z^2}$ is the same as the centrifugal contribution in the case of a two-dimensional harmonic Lagrangian, written in polar coordinates. Such a case can be solved in arbitrary dimension. The explicit expression and the algebraic and analytical manipulations of these rather cumbersome mathematical aspects are given in Appendix C.

The solution of the path integral (119) can be used to calculate the zero-coupon price by Eq. (83). The final result, quoted in the literature (see, for example, [19]), can be written as

$$P(r_t, t, T) = A(t, T) e^{-B(t, T) r_t}, \quad (124)$$

where the functions $B(t, T)$, and $A(t, T)$ are given in the Eqs. (C9), and (C10). We stress further that in this case the anomalous $1/z^2$ dependence masks the underlying quadratic problem. Once this connection is understood the solution is straightforward.

The expression above represents a closed-form solution for the zero-coupon price in the CIR model. Obviously, the explicit knowledge of the Green function allows the evaluation of more complex functionals such as, for example, cap or floor prices.

Moreover, the method can be applied to coupled SDEs. The general theorems on quadratic Lagrangians allow to solve by diagonalization a multi-dimensional coupled problem, the only constraint is the absence of higher order terms ($O(x^3)$ or larger).

Finally, this treatment can be extended to more complex problems with non-quadratic Lagrangians. The scientific literature suggests a variety of approximation techniques: for example, the perturbative, and the saddle point methods.

In conclusion, the path integral approach, besides other analytical methods (see, for example, [20]), represents a powerful tool of analysis.

VI. CONCLUSIONS

We have described the path integral method as an alternative approach to find the solution of stochastic equations. We have shown that the method can be used in general cases, and that it is equivalent to the formulation in terms of SDE and PDE equations. The method is suitable in particular for defining functional mean values and path dependent problems. However, an exact analytical treatment is possible only in a few cases. A constraint necessary to get analytical solutions is the quadratic form of the Lagrangian describing the stochastic process. We have discussed in general this case, and the problem of the coupling between the stochastic variable and its derivative. Moreover, we have given the explicit solution for some important one-dimensional problems, and briefly discussed the generalization to quadratic multi-dimensional cases. Finally, we point out that the path integral treatment can be extended to non-quadratic cases by approximate analytical techniques, or by numerical methods. The numerical approach will be described in a forthcoming paper (see also Ref. [21]).

APPENDIX A: EQUIVALENCE BETWEEN THE PATH INTEGRAL APPROACH AND THE PARTIAL DERIVATIVE EQUATIONS

We want to show that the short-time transition probability (13) gives rise to the Fokker-Planck equation (10) corresponding to the differential stochastic equation (5). Let us write the following identity

$$\rho(x, T + \Delta t | y, t) = \int_{-\infty}^{+\infty} dz \rho(x, T + \Delta t | z, T) \rho(z, T | y, t). \quad (\text{A1})$$

If we use the approximate expression (13), and employing the identity $z = x + (z - x) = x + \eta$, we obtain

$$\rho(x, T + \Delta t | y, t) \simeq \int_{-\infty}^{+\infty} d\eta \frac{1}{\sqrt{2\pi\Delta t\sigma^2}} \exp \left\{ \frac{(-\eta - A(x + \eta, T) \Delta t)^2}{2\Delta t\sigma^2} \right\} \rho(x + \eta, T | y, t), \quad (\text{A2})$$

where, for the sake of simplicity, we have taken $\sigma = \text{constant}$. If we recall that $\eta \sim O(\sqrt{\Delta t})$, and expand the above expression in η up to $O(\Delta t)$, we get

$$\begin{aligned} \rho(x, T + \Delta t | y, t) &\simeq \int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{2\pi\Delta t\sigma^2}} \exp \left\{ -\frac{\eta^2}{2\Delta t\sigma^2} \right\} \\ &\times \left\{ 1 - \frac{\eta}{\sigma^2} A(x, T) - \frac{\eta^2}{\sigma^2} \frac{\partial A(x, T)}{\partial x} + \frac{\eta^2}{\sigma^4} A(x, T)^2 \right\} \left\{ 1 - \frac{\Delta t A(x, T)^2}{2\sigma^2} \right\} \\ &\times \left\{ \rho(x, T | y, t) + \eta \frac{\partial}{\partial x} \rho(x, T | y, t) + \frac{\eta^2}{2} \frac{\partial^2}{\partial x^2} \rho(x, T | y, t) \right\}. \end{aligned} \quad (\text{A3})$$

Then, by performing all calculations, we find

$$\rho(x, T + \Delta t | y, t) \simeq \rho(x, T | y, t) - \Delta t \left[\frac{\partial A(x, T) \rho(x, T | y, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 \sigma^2 \rho(x, T | y, t)}{\partial x^2} \right]. \quad (\text{A4})$$

Since the L.H.S. of this equation can be also written, up to $O(\Delta t)$, as

$$\rho(x, T + \Delta t | y, t) \simeq \rho(x, T | y, t) + \Delta t \frac{\partial}{\partial T} \rho(x, T | y, t), \quad (\text{A5})$$

by collecting all pieces together, we obtain the Fokker-Planck equation.

The general case where there is an explicit dependence of the function, σ , on x , and T , is more complex. We must expand the equation up to $O(\eta^8)$, and $O(\Delta t^4)$. However after several cumbersome calculation we obtain the final result

$$\frac{\partial \rho(x, T | y, t)}{\partial T} = - \frac{\partial A(x, t) \rho(x, T | y, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 \sigma^2(x, t) \rho(x, T | y, t)}{\partial x^2}, \quad (\text{A6})$$

which coincides with the general Fokker-Planck equation (10).

APPENDIX B: EXPONENT EXPANSION

In this appendix we describe a procedure to obtain an expression for the short-time transition probability which is correct up to any given order [22]. Here, for the sake of simplicity, we consider the one-dimensional case, but the formalism can be easily extended to arbitrary dimensions. Moreover we take $\sigma = \text{constant}$, and $A(x, \tau) = a(x)$.

The starting point is that the transition probability must satisfy the Fokker-Planck equation (10). Therefore, if we make the following ansatz,

$$\rho(x, t + \Delta t | y, t) = \frac{1}{\sqrt{2\pi\Delta t\sigma^2}} \exp \left\{ \frac{-(x-y)^2}{2\Delta t\sigma^2} - f(x, y, \Delta t) \right\}, \quad (\text{B1})$$

the function, $f(x, y, \Delta t)$, must satisfy the equation

$$\begin{aligned} \frac{\partial}{\partial \Delta t} f(x, y, \Delta t) = & -a(x) \frac{\partial}{\partial x} f(x, y, \Delta t) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} f(x, y, \Delta t) \\ & - \frac{\partial}{\partial x} a(x) + \frac{(x-y)}{\Delta t} \frac{\partial}{\partial x} f(x, y, \Delta t) - \frac{(x-y)}{\Delta t \sigma^2} a(x). \end{aligned} \quad (\text{B2})$$

Let us now expand $f(x, y, \Delta t)$ in powers of Δt ,

$$f(x, y, \Delta t) = f_0(x, y) + \Delta t f_1(x, y) + \Delta t^2 f_2(x, y) + \Delta t^3 f_3(x, y) + \dots; \quad (\text{B3})$$

then, by substituting in (B2), we get the following set of recursive equations,

$$\begin{cases} \partial_x f_0(x, y) = & -\frac{1}{\sigma^2} a(x) \\ f_1(x, y) = & -(x-y) \partial_x f_1(x, y) + \frac{1}{2} \partial_x a(x) + \frac{a(x)^2}{2\sigma^2} \\ 2 f_2(x, y) = & -(x-y) \partial_x f_2(x, y) + \frac{1}{2} \sigma^2 \partial_x^2 f_1(x, y) \\ 3 f_3(x, y) = & -(x-y) \partial_x f_3(x, y) + \frac{1}{2} \sigma^2 \partial_x^2 f_2(x, y) - \frac{1}{2} \sigma^2 (\partial_x f_1(x, y))^2 \\ 4 f_4(x, y) = & -(x-y) \partial_x f_4(x, y) + \frac{1}{2} \sigma^2 \partial_x^2 f_3(x, y) - \frac{1}{2} \sigma^2 \partial_x f_1(x, y) \partial_x f_2(x, y) \\ 5 f_5(x, y) = & \dots \end{cases} \quad (\text{B4})$$

The first equation is decoupled by the others and defines an overall phase factor. The term $f_0(x, y)$ is related to the integral of $a(x)$, and it is just the phase factor (28). The second equation gives the first order approximation in Δt ; its value can be used as input for the successive equation, and so on. Therefore, by a simple iteration, we can compute the function $f(x, y, \Delta t)$ up to the required order. In general, the structure of these equations is

$$n f_n(x, y) = -(x-y) \partial_x f_n(x, y) + W_n(x, y), \quad (\text{B5})$$

where $W_n(x, y)$ depends on the functions, $f_1(x, y), \dots, f_{n-1}(x, y)$. The solution is given by

$$f_n(x, y) = \int_0^1 d\xi \xi^{n-1} W_n(y + \xi(x-y), y). \quad (\text{B6})$$

Actually the integral appearing in Eq. (B6) can be rather complicated, but the problem could be simplified by a further expansion in $\Delta x = x - y$ [23].

APPENDIX C: EXPLICIT CALCULATION FOR THE CIR MODEL

The path integral (119) has an explicit solution given by (see, for example, [16])

$$\begin{aligned} G(z_T, T | z_t, t) = & \frac{2}{z_T} \exp \left\{ \frac{\Delta \phi}{\sigma^2} \right\} \frac{\gamma \sqrt{z_T z_t}}{2\sigma^2 \sinh \left[\frac{\gamma}{2} (T-t) \right]} \\ & \times \exp \left[-\frac{\gamma}{4\sigma^2} (z_T^2 + z_t^2) \coth \left(\frac{\gamma}{2} (T-t) \right) \right] I_\mu \left(\frac{z_T z_t \gamma}{2\sigma^2 \sinh \left(\frac{\gamma}{2} (T-t) \right)} \right) e^{a^2 b(T-t)/\sigma^2}, \end{aligned} \quad (\text{C1})$$

with $I_\mu(x)$ the modified Bessel function of index μ ,

$$\gamma = \sqrt{a^2 + 2\sigma^2}, \quad (\text{C2})$$

and

$$\mu = \frac{1}{2} \sqrt{1 + \frac{8c_3}{\sigma^2}} = \frac{2ab}{\sigma^2} - 1. \quad (\text{C3})$$

In order to get the formula for the zero-coupon price this expression must be integrated over the final variable, r_T :

$$P(r_t, t, T) = \int_0^{+\infty} dr_T G(z_T, T \mid z_t, t) = \int_0^{+\infty} dz_T \frac{z_T}{2} G(z_T, T \mid z_t, t). \quad (\text{C4})$$

The integral looks rather formidable, however it can be handled thanks to the integral

$$\int_0^{+\infty} e^{-\alpha z^2} z^{\mu+1} I_\mu(\beta z) dz = \frac{\beta^\mu}{(2\alpha)^{\mu+1}} \exp\left[\frac{\beta^2}{4\alpha}\right]. \quad (\text{C5})$$

In the equation above the parameter α , and β are given by

$$\alpha = \frac{\gamma}{4\sigma^2} \coth\left[\frac{\gamma}{2}(T-t)\right] + \frac{a}{4\sigma^2}, \quad (\text{C6})$$

and

$$\beta = \frac{\gamma z_t}{2\sigma^2 \sinh\left[\frac{\gamma}{2}(T-t)\right]}. \quad (\text{C7})$$

Therefore, by performing the integration, and inserting the original variable, r_t , we obtain the expression

$$\begin{aligned} P(r_t, t, T) = & \exp\left[\frac{\beta^2}{4\alpha}\right] \exp\left\{\frac{a r_t}{\sigma^2} - \frac{\gamma r_t}{\sigma^2} \coth\left[\frac{\gamma}{2}(T-t)\right]\right\} \\ & \times \left\{\frac{\gamma}{4\sigma^2 \alpha \sinh\left[\frac{\gamma}{2}(T-t)\right]}\right\}^{\mu+1} \exp\left[\frac{a^2 b}{\sigma^2}(T-t)\right] \end{aligned} \quad (\text{C8})$$

Note that the dependence of $P(r_t, t, T)$ on r_t is only due to the exponential, $\exp[-B(t, T) r_t]$, where

$$B(t, T) = \frac{-\gamma^2}{4\alpha\sigma^4 \sinh^2\left[\frac{\gamma}{2}(T-t)\right]} - \frac{a}{\sigma^2} + \frac{\gamma}{\sigma^2} \coth\left[\frac{\gamma}{2}(T-t)\right] = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}. \quad (\text{C9})$$

Finally, if we define $A(t, T)$ by

$$\begin{aligned} A(t, T) = & \left\{\frac{\gamma}{4\sigma^2 \alpha \sinh\left[\frac{\gamma}{2}(T-t)\right]}\right\}^{\mu+1} \exp\left[\frac{a^2 b}{\sigma^2}(T-t)\right] \\ = & \left[\frac{2\gamma e^{\gamma(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2ab/\sigma^2} e^{a^2 b(T-t)/\sigma^2}, \end{aligned} \quad (\text{C10})$$

we can cast the expression for the zero-coupon price, $P(r_t, t, T)$, in the form given in Eq. (124).

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